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Random field Ising model in the Bethe–Peierls approximation

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Abstract. The random field Ising model is solved numerically in the Bethe-Peierls approximation. For a model with a two-peak δ-distribution, the transition is first order at low temperatures and second order at high temperatures, and the tricritical point appears as an inflection point of the transition curve. The behaviour at low temperatures is analysed analytically as a function of the coordination number, and compared with the mean-field prediction.

1. Introduction

The effects of a random quenched magnetic field upon the critical behaviour near a ferromagnetic phase transition has attracted considerable attention in recent years (for a general review see Imry 1984). The simplest way to treat this problem is by the mean-field approximation. This was carried out by Schneider and Pytte (1977) for a model with a Gaussian distribution of fields and led to a line of second-order transitions. Aharony (1978) considered more general distributions. In particular he showed that for a two-peak δ-distribution there is a tricritical point and that at sufficiently low temperatures the phase transition becomes first order.

The Bethe-Peierls approximation is an improvement over the mean-field approximation, since it takes into account specific short-range order. In a pure system (i.e., in the absence of the random field) the equations of the Bethe-Peierls approximation describe exactly the Bethe lattice (Bethe 1935). In the presence of the random field this is not the case.

The random field Ising ferromagnet of a Bethe lattice was investigated by Bruinsma (1984) and Entin-Wohlman and Domb (1984). Bruinsma considered in detail the ground state at $T=0$ and argued that at $T=0$ the critical behaviour is mean field like. Entin-Wohlman and Domb have found that an iterative solution of the recursion relation of the Bethe lattice leads to the same low-temperature series as derived from the linked-cluster expansion (Domb 1970). They have shown that such an expansion is possible if the random field distribution is bounded to finite values of the random field. When it is not so (e.g., a Gaussian distribution) the lowest energy state will contain overturned spins and will not correspond to a complete ferromagnetic order. The analysis of the low-temperature series for a two-peak δ-distribution has indicated that at sufficiently low temperature the transition is first order, while close enough to

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the transition temperature T_c of the pure system it is second order (Entin-Wohlman and Domb 1984).

In order to explore the whole temperature range ($0 < T < T_c$) we consider in this paper the Bethe-Peierls approximation for the random field Ising model with a two-peak δ-distribution. Unlike the case of the Bethe lattice, the equations of the Bethe-Peierls approximation can be exactly solved. We find that the transition line has an inflection point at the tricritical point. At very low temperatures and in the temperature range close to T_c the results agree with those of the Bethe lattice (Entin-Wohlman and Domb 1984). We discuss the large coordination number limit and compare it with the predictions of the mean-field treatment (Aharony 1978).

2. The Bethe–Peierls approximation

In the Bethe-Peierls approximation one retains the interaction of a single spin σ_0 and its surrounding q-neighbours, and takes into account the other spins only by way of a molecular field H' which acts upon the q-neighbouring spins. Thus the Hamiltonian of the central spin σ_0 and its q nearest neighbours reads

$$\mathscr{H} = -J \sum_{j=1}^{q} \sigma_0\sigma_j - H_0\sigma_0 - \sum_{j=1}^{q} (H_j + H')\sigma_j. \tag{1}$$

Here the first term describes the Ising interaction between σ_0 and its nearest neighbours and H_l is the random field acting upon σ_l ($l = 0, 1, \ldots, q$). The field H' that the neighbours of the central spin feel is not only taken as independent of the other spins, but also independent of the random fields on the other sites. This is an approximation even on a Bethe lattice. Thus, contrary to the pure case, the Bethe approximation for the random field Ising model is not exact on the Bethe lattice. The molecular field H' is determined by the condition that the average of σ_0 is equal to the average of σ_j ($j = 1, 2, \ldots q$). The average of σ_0 is given by

$$\langle\sigma_0\rangle = [(z_+ - z_-)/(z_+ + z_-)]_{\mathrm{av}}, \tag{2}$$

where

$$z_\pm = \mathrm{e}^{\pm\beta H_0} \prod_{j=1}^{q} 2\cosh(\pm\beta J + \beta H' + \beta H_j), \tag{3}$$

and $[\ ]_{\mathrm{av}}$ denotes an average upon the random field distribution. The average of σ_j is given by

$$\langle\sigma_j\rangle = \frac{1}{q}\left[\frac{1}{z_+ + z_-}\left(z_+ \sum_{j=1}^{q} \tanh(\beta J + \beta H' + \beta H_j) + z_- \sum_{j=1}^{q} \tanh(-\beta J + \beta H' + \beta H_j)\right)\right]_{\mathrm{av}}. \tag{4}$$

Equating equation (2) to equation (4) yields an equation for the molecular field H'—this is the self-consistency equation of the Bethe-Peierls approximation. Inserting the solution into either (2) or (4) gives the magnetisation (per spin) as a function of temperature and the random field distribution.

When the random field obeys a two-peak δ-distribution

$$P(H) = \tfrac{1}{2}[\delta(H-h) + \delta(H+h)], \tag{5}$$

it is straightforward to follow numerically the procedure outlined above. Introducing the variable τ

$$\tau = \tanh \beta H' \tag{6}$$

we solve the equation

$$\langle \sigma_0 \rangle - \langle \sigma_j \rangle = 0 \tag{7}$$

to find τ as a function of h/J and $t = T/T_c$, where T_c is the pure system ($h = 0$) transition temperature. Figure 1 depicts the LHS of equation (7), as a function of τ, for the case $q = 3$. The solution $\tau = 0$ corresponds to zero magnetisation ($\langle \sigma_0 \rangle = 0$). At high enough values of h/J, only this solution survives. The behaviour of the other solution is rather interesting. Starting at high temperatures (close to T_c) we see that the solution approaches zero as h/J increases (figure 1(a)). However, at lower temperatures the function changes its behaviour (figure 1(b)). A third solution appears

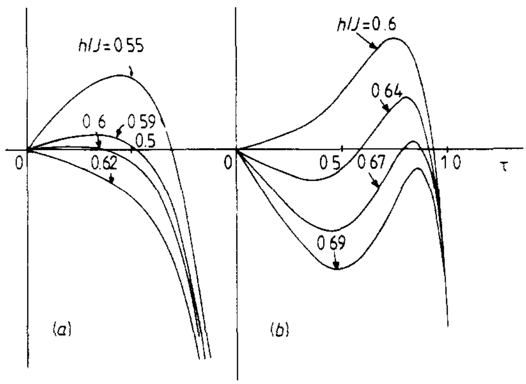

Figure 1. The LHS of equation (7) for various values of h/J in the case $q = 3$. (a) $T/T_c = 0.7$, (b) $T/T_c = 0.53$.

(the intermediate solution in figure 1(b)) which is of no interest (for this solution τ increases with h/J, leading to an increase in the magnetisation as h/J increases). The rightmost solution decreases as h/J increases, until it suddenly disappears. When this solution is inserted into equation (2) it causes the magnetisation to jump from some finite value to zero, and thus to go through a first-order transition. We portray the behaviour of the magnetisation as a function of h/J for various values of t in figure 2.

We have checked the behaviour of the free energy as a function of τ (equation (6)) and the results are depicted in figure 3 for the case $q = 3$. The free energy is given by

$$F = -(1/\beta)[\ln(z_+ + z_-)]_{\text{av}}, \tag{8}$$

and it is a monotonically decreasing function of τ. Therefore the highest possible solution for τ corresponds to the lowest possible free energy.

By scanning the (h/J)–t plane we have obtained the transition line above which the magnetisation is zero. We have found that this line has an inflection point where the transition changes from second order (high-temperature side) to first order. This

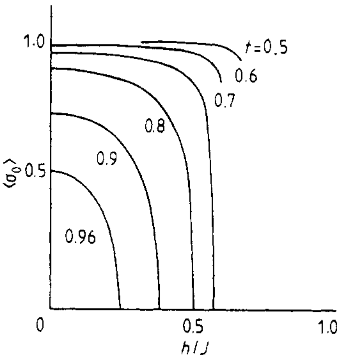

Figure 2. The magnetisation $\langle\sigma_0\rangle$ for the case $q=3$.

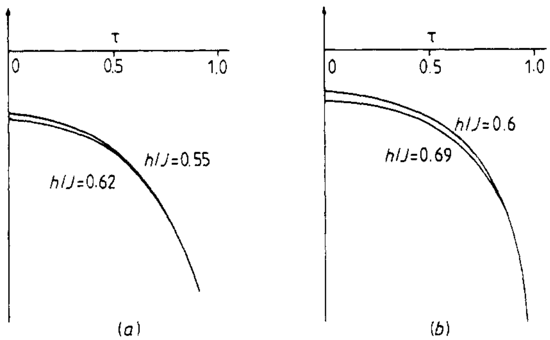

Figure 3. The free energy as a function of τ for the case $q=3$. (a) $T/T_c=0.7$, (b) $T/T_c=0.53$ (the y-axis is in arbitrary units).

is shown in figure 4 for the cases $q=3$ and $q=6$. One sees that the first-order transition line approaches the point $t=0$ as a straight line. Moreover, as q increases, this line becomes steeper. The equation of this line can be found by examining equation (7) at low temperatures. To this end we introduce the notations

$$z=e^{-2\beta J}, \qquad \theta=e^{-2\beta H'}, \qquad \lambda_j=e^{-2\beta H_j}. \tag{9a, b, c}$$

Using these notations in equation (7) we get an equation for θ

$$\left[\frac{\theta\lambda_1-A}{1+z\theta\lambda_1+(z+\theta\lambda_1)A}\right]_{\text{av}}=0, \tag{10}$$

where

$$A=\lambda_0\prod_{j=2}^{q}(z+\theta\lambda_j)/(1+z\theta\lambda_j). \tag{11}$$

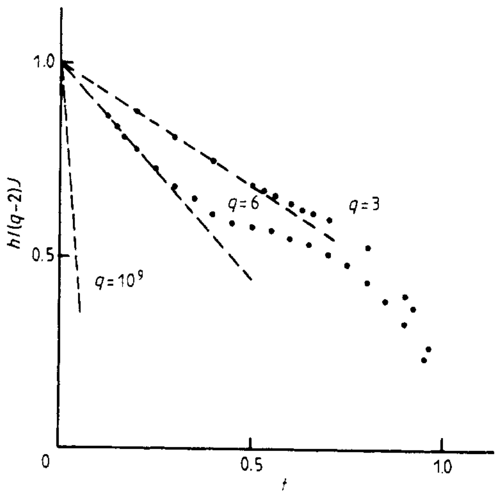

Figure 4. The transition lines for the cases $q=3$ and $q=6$, represented by the dotted curves (the dots are the result of the numerical analysis). The inflection points ($t=0.63$, $h/J=0.62$ for $q=3$, $t=0.45$, $h/J=2.35$ for $q=6$) mark the change in the order of the transition from second (high-temperature side) to first-order transition. The broken lines represent equation (17) (see text).

To lowest order in z equation (10) yields

$$\theta = [z + \theta(1 - z^2) m_1]^{q-1}, \tag{12}$$

where

$$m_1 = [\lambda_j]_{\rm av}. \tag{13}$$

Equation (12) is the same as that obtained in the analysis of the low-temperature series of the Bethe lattice (Entin-Wohlman and Domb 1984). Its derivation is based upon the requirement that $\theta m_1 < z$, i.e., $z^{q-2} m_1 < 1$. It has been shown (Entin-Wohlman and Domb 1984) that this condition is satisfied for a two-peak δ-distribution or, more generally, for any bounded distribution such that $h < J(q-2)$ where $\pm h$ are the distribution bounds. It is not satisfied for an unbounded distribution, e.g., a Gaussian distribution. Inspection of (12) reveals that it has a solution provided that

$$\theta \leqslant [m_1(q-1)(1-z^2)]^{(1-q)/(q-2)}. \tag{14}$$

Equations (12), (13) and (14) yield a line in the h/J-t plane

$$z(q-1)/(q-2) = [m_1(1-z^2)(q-1)]^{1/(2-q)}. \tag{15}$$

Above this line equation (12) has no solution. Returning to the original equation (10) for θ, we see that $\theta = 1$ is also a possible solution, provided that the random field distribution $P(H)$ is symmetric in H. The magnetisation is given (from equations (2) and (9–11)) by

$$\langle \sigma_0 \rangle = \left[\frac{1 - \theta\lambda_1 A}{1 + z\theta\lambda_1 + (z + \theta\lambda_1)A} \right]_{\rm av}. \tag{16}$$

This vanishes for $\theta = 1$ and a symmetric distribution. On the other hand, $\langle \sigma_0 \rangle$ is finite below the line given by equation (15). Thus we conclude that (15) gives the first-order transition line at low temperatures.

When the random field obeys the distribution (5), equation (15) gives

$$\frac{h}{(q-2)J} = 1 - \left[\ln\left(\frac{q-2}{q-1} \left(\frac{2}{q-1} \right)^{1/(q-2)} \right) \Big/ \ln\left(\frac{q-2}{q} \right) \right] t, \tag{17}$$

where we have used $2\beta_c J = \ln[q/(q-2)]$. The lines obtained from (17) are shown in figure 4. In the cases $q = 3$ and $q = 6$, they coincide with the transition lines found in the numerical procedure. When q becomes very large, the line becomes steeper (e.g., $q = 10^9$ in figure 4). The large-q limit is expected to yield the mean-field behaviour. We have therefore re-examined the mean-field treatment with a two-peak δ-distribution (Aharony 1978) in the $T = 0$ limit. Aharony finds that the magnetisation M is the solution of the equation

$$M = [\tanh \beta(qJM + H_j)]_{\rm av}. \tag{18}$$

Averaging over the random field with the distribution (5)

$$M = \tfrac{1}{2}[\tanh \beta(qJM + h) + \tanh \beta(qJM - h)]. \tag{19}$$

In the $\beta \to \infty$ limit, the RHS yields zero unless $h < qJM$, in which case it gives 1. Thus on the $T = 0$ line the magnetisation is one below $h/qJ = 1$ and zero above, in accordance with the Bethe–Peierls approximation in the high-q limit.

On the other hand, when the random field distribution is not bounded we cannot expand (10) for small θ at low temperatures. It follows that the solution of (10) is $\theta = 1$ which corresponds to $\langle\sigma_0\rangle = 0$. Thus, for unbounded distributions, the mean-field treatment is in disagreement with the Bethe-Peierls approximation in the low-temperature range.

3. Discussion

We have analysed the random field Ising model in the Bethe-Peierls approximation. In particular, we have discussed the first-order transition line at low temperatures. The discussion is confined to symmetric distributions. When the distribution is non-symmetric, a zero magnetisation is never a solution (unless one adds a uniform field). Our results agree with the series expansion analysis of the Bethe lattice (Entin-Wohlman and Domb 1984).

The main feature of the transition curve for a two-peak δ-distribution is the inflection point at the tricritical point. Unfortunately, we have not succeeded in analysing this point (e.g., as a function of q) analytically. It is quite straightforward to obtain the equation for the second-order transition line for small random fields and the vicinity of T_c. The result is

$$\omega(q-1)\{1 - M_2[1 - \omega + (q-2)\omega^2]\} = 1, \tag{20}$$

where

$$\omega = \tanh \beta J, \qquad M_2 = [\tanh^2 \beta H_j]_{\mathrm{av}}. \tag{21a, b}$$

However, this is valid only in a small region close to T_c. The inflection point occurs far away from there (see figure 4), and expansions around it are too complicated.

In a pure system, the Bethe-Peierls approximation describes the main features of the phase transition fairly well. We hope to deal with its relevance to the behaviour of the random field Ising model on standard lattices in the near future.

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